MATH2118 Lecture Notes

Further Engineering Mathematics C

Ordinary Differential Equations

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Contents

1 Introduction

A *differential equation* is an equation containing the derivatives (or differentials) of one or more dependent variables, with respect to one or more independent variables.

EXAMPLE

- Current $i(t)$ flowing in a circuit with applied e.m.f. $E(t)$:

$$L\frac{di}{dt} + Ri = E(t)$$

- Angular displacement $\theta(t)$ of a rigid body pendulum:

$$I\frac{d^2\theta}{dt^2} + mgh \cdot \sin\theta = 0$$

- Mechanical vibrations:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Differential equations are classified according to *type, order* and *linearity.* If an equation contains any ordinary derivatives of one or more dependent variables, with respect to a single independent variable, it is then said to be an *ordinary differential equation.* The order of the highest derivative in a differential equations is called the *order of the equation.*

A differential equation is said to be *linear* if it has the form

$$\boxed{a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).}$$

Note the power of each term involving y is one, and each coefficient a_n depends only on the independent variable x. An equation that is not linear is said to be *nonlinear.*

EXAMPLE

(1) $x\,dy + y\,dx = 0$ $\longleftarrow$ first-order, linear

(2) $y'' - 2y' + y = 0$ $\longleftarrow$ second-order, linear

(3) $yy'' - 2y' = x$ $\longleftarrow$ second-order, nonlinear

(4) $\dfrac{d^3y}{dx^3} + y^2 = 0$ $\longleftarrow$ third-order, nonlinear

2 First-Order Differential Equations

The most general first-order differential equation has the form

$$\boxed{f\big(x, y, y'\big) = 0.}$$

Two common classes of such equations are

- Separable type:

$$\boxed{\frac{dy}{dx} = F(x) \cdot G(y)}$$

- First-order linear type:

$$\boxed{\frac{dy}{dx} + p(x)y = q(x)}$$

2.1 Separable Differential Equations

The differential $\dfrac{dy}{dx} = f(x, y)$ is said to be *separable* if the right-hand-side terms can be written as a product of two factors:

$$\begin{aligned} \frac{dy}{dx} &= f(x,y) = F(x) \cdot G(y) \\ \Rightarrow \quad \frac{dy}{G(y)} &= F(x)\,dx \quad \text{provided } G(y) \neq 0 \\ \Rightarrow \int \frac{dy}{G(y)} &= \int F(x)\,dx \end{aligned}$$

Here $F(x)$ contains x only, and $G(y)$ contains y only.

EXAMPLE

Solve $yy' = x$.

SOLUTION

$$\begin{aligned} \frac{dy}{dx} &= \frac{x}{y} \quad \text{provided } y \neq 0 \\ \Rightarrow \; y\,dy &= x\,dx \end{aligned}$$

Integrating both sides:

$$\begin{aligned} \int y\,dy &= \int x\,dx \\ \Rightarrow \quad \frac{1}{2}y^2 &= \frac{1}{2}x^2 + c_0 \\ \Rightarrow \quad y^2 &= x^2 + c \qquad (\text{putting } c = 2c_0) \end{aligned}$$

This is the *general solution*, since c is an arbitrary constant.

NOTE:

- $*$ first-order ODE produces 1 arbitrary constant
- $*$ second-order ODE produces 2 arbitrary constants
- $*$ n^{th}-order ODE produces n arbitrary constants

EXAMPLE

Solve $y' + 2y = 4$ subject to $y(0) = 0$.

SOLUTION

$$\begin{aligned}
\frac{dy}{dx} &= 4 - 2y \\
\Rightarrow \quad \int \frac{dy}{4-2y} &= \int dx \\
\Rightarrow \quad -\frac{1}{2}\ln|4-2y| &= x + c \\
\Rightarrow \quad \ln|4-2y| &= -2(x+c) \\
\Rightarrow \quad 4 - 2y &= e^{-2(x+c)} \quad \text{since } e^{-(x+c)} > 0 \\
\Rightarrow \quad y &= 2 - \frac{1}{2}e^{-2(x+c)} \\
&= 2 + Ae^{-2x} \quad \text{with } A = -\frac{1}{2}e^{-2c}
\end{aligned}$$

Applying the condition $y(0) = 0$:

$$0 = 2 + Ae^{0} \quad \Rightarrow A = -2$$

Thus, the particular solution is

$$y(x) = 2\big(1 - e^{-2x}\big).$$

EXAMPLE

The mass M of a radioactive substance is initially 10 g, and 20 years later its mass is 9.6 g. Given that the rate of decay of a radioactive substance is proportional to the mass of that substance present at any time t, in how many years will the mass be halved (half-life) ?

Governing equation:

$$\frac{-dM}{dt} \propto M \qquad (M \text{ in grams and } t \text{ in years})$$

$$\Rightarrow \frac{dM}{dt} = -kM \quad \text{where } k \text{ is a proportionality constant.}$$

Separating the variables and integrating both sides:

$$\int \frac{dM}{M} = -k \int dt$$

$$\Rightarrow \quad \ln|M| = -kt + c$$

$$\Rightarrow \quad M(t) = e^{-kt+c}$$

$$= Ae^{-kt} \quad \text{where } A = e^c = \text{constant.}$$

Two constants (A, k) require two known conditions:

when $t = 0,\ M = 10$: $\quad 10 = Ae^0 = A$

$$\Rightarrow M = 10e^{-kt}$$

when $t = 20,\ M = 9.6$: $\quad 9.6 = 10e^{-20k}$

$$\Rightarrow \ln 0.96 = -20k$$

$$\Rightarrow \quad -k = 0.05 \ln 0.96$$

$$\Rightarrow \quad M(t) = 10e^{0.05t \ln 0.96}$$

when $M = 5,\ t = ?$: $\quad 5 = 10e^{0.05t \ln 0.96}$

$$\Rightarrow \ln 0.5 = 0.05t \ln 0.96$$

$$\Rightarrow \quad t = \frac{20 \ln 0.5}{\ln 0.96}$$

$$\approx 340 \text{ years.}$$

2.2 First-Order Linear Differential Equations

General form of n-order linear differential equation:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

When $n = 1$, we obtain the first-order linear differential equation:

$$\boxed{a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).}$$

Rewriting this equation as

$$\frac{dy}{dx} + \underbrace{\frac{a_0(x)}{a_1(x)}}_{=p(x)} y = \underbrace{\frac{g(x)}{a_1(x)}}_{=q(x)} \qquad \text{provided } a_1(x) \neq 0$$

$$\Rightarrow \frac{dy}{dx} + p(x)y = q(x) \qquad \longleftarrow \text{linear in both } y \text{ \& } y',\ \text{coefficient of } y' \text{ is 1.} \tag{1}$$

To solve this differential equation, we multiply both sides of (1) by some function $I(x)$, such that the left-hand-side term becomes an exact derivative,

$$I(x)\frac{dy}{dx} + I(x)p(x)y = I(x)q(x), \tag{2}$$

and

$$\frac{d}{dx}\big(Iy\big) = I\frac{dy}{dx} + y\frac{dI}{dx}\,. \tag{3}$$

We want the left-hand-side of (2) same as the left-hand-side of (3):

$$\begin{aligned}
\frac{d}{dx}\big(Iy\big) &= I\frac{dy}{dx} + y\frac{dI}{dx} = I\frac{dy}{dx} + Ipy \\
\Rightarrow \quad \frac{dI}{dx} &= Ip \\
\Rightarrow \quad \frac{dI}{I} &= p\,dx
\end{aligned}$$

Thus,

$$\begin{aligned}
\int \frac{dI}{I(x)} &= \int p(x)\,dx \\
\Rightarrow \ \ln\big|I(x)\big| &= \int p(x)\,dx \\
\Rightarrow \quad \big|I(x)\big| &= e^{\int p(x)\,dx} \\
\Rightarrow \quad I(x) &= e^{\int p(x)\,dx} \quad \text{since } e^{\int p(x)\,dx} > 0 \text{ for all } x \qquad (4) \\
&= \textit{integrating factor}
\end{aligned}$$

With $I(x)$ given by (4), Equation (1) can then be solved as

$$\begin{aligned}
\frac{d}{dx}\Big[I(x)y(x)\Big] &= I(x)q(x) \\
\Rightarrow \ y(x) &= \frac{1}{I(x)}\int I(x)q(x)\,dx \qquad \longleftarrow \text{coefficient of } y' \text{ is } 1
\end{aligned}$$

▌EXAMPLE

Solve $xy' - 4y = x^6 e^x$.

SOLUTION

Rewrite this equation such that the coefficient of y' is 1:

$$\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x \qquad \longleftarrow p(x) = -4/x,\ q(x) = x^5 e^x$$

Integrating factor $I(x)$:

$$\begin{aligned}
I(x) &= e^{\int p(x)\,dx} \\
&= e^{-\int \frac{4}{x}\,dx}
\end{aligned}$$

$$
\begin{aligned}
&= e^{-4\ln|x|} \\
&= e^{\ln|x|^{-4}} \\
&= \frac{1}{x^4} \quad \text{provided } x \neq 0.
\end{aligned}
$$

Thus, $\dfrac{d}{dx}(Iy) = Iq$ becomes

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{y}{x^4}\right) &= \frac{x^5 e^x}{x^4} \\
&= xe^x \\
\Rightarrow \quad \frac{y}{x^4} &= \int xe^x \, dx \\
\Rightarrow \quad y &= x^4 \int xe^x \, dx
\end{aligned}
$$

Applying integration by parts method to the above integral by letting $u = x$, $du = dx$, $dv = e^x \, dx$ and $v = e^x$, we obtain

$$
\begin{aligned}
\int xe^x \, dx &= uv - \int v \, du \\
&= xe^x - \int e^x \, dx \\
&= xe^x - e^x + c.
\end{aligned}
$$

General solution:

$$
\begin{aligned}
y(x) &= x^4\big(xe^x - e^x + c\big) \\
&= x^4(x-1)e^x + cx^4.
\end{aligned}
$$

EXAMPLE

Salt solution containing 2 g/lt of salt flows into a tank initially filled with 50 lt of water containing 10 g of salt. If the solution enters the tank at 5 lt/min, the concentration is kept uniform by stirring, and the mixture flows out at the same rate, find the amount of salt in the tank after 10 mins.

SOLUTION

Suppose there are Q g of salt in the tank after t mins. Since 5 lt of salt solution enter and leave the tank each minute, the tank will contain 50 lt of solution at any time t. Therefore, the solution concentration will be $\frac{Q}{50}$ g/lt, and since it flows out at 5 lt/min, the rate of outflow is $\frac{Q}{50} \times 5 = \frac{Q}{10}$ g/min, while the inflow rate is 2 g/lt $\times$ 5 lt/min = 10 g/min.

Rate of increase of $Q(t)$:

$$\begin{aligned} \frac{dQ}{dt} &= \text{inflow rate} - \text{outflow rate} \\ &= 10 - \frac{Q}{10} \\ \Rightarrow \quad \frac{dQ}{dt} + \frac{Q}{10} &= 10 \end{aligned}$$

Solve for $Q(t)$:

$$\begin{aligned} I &= e^{\int \frac{1}{10}\,dt} = e^{t/10} \\ \Rightarrow \quad \frac{d}{dt}\big(e^{t/10}Q\big) &= 10e^{t/10} \\ \Rightarrow \quad e^{t/10}Q &= 10\int e^{t/10}\,dt \\ &= 100e^{t/10} + c \\ \Rightarrow \quad Q &= 100 + ce^{-t/10} \end{aligned}$$

When $t = 0$, $Q = 10\,\text{g}$:

$$10 = 100 + ce^{0} \qquad \Rightarrow c = -90.$$

Thus,

$$Q(t) = 100 - 90e^{-t/10}.$$

After 10 mins ($t = 10$):

$$\begin{aligned} Q(t=10) &= 100 - 90e^{-10/10} \\ &= 100 - 90/e \\ &\approx 66.9\,\text{g of salt.} \end{aligned}$$

3 Second-Order Differential Equations

A n^{th}-order linear differential equation,

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is said to be *nonhomogeneous* if $g(x) \neq 0$ for some x values. If $g(x) = 0$ for every x, then the differential equation is said to be *homogeneous*. We only concerned with finding solutions of second-order linear differential equation with real constant coefficients,

$$\boxed{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x)}$$

where a, b, c are real constants.

3.1 Homogeneous Equation

Homogeneous equation:

$$\boxed{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0}$$

Let $y_1(x)$ and $y_2(x)$ be solutions of this equation. According to the *superposition principle*, the linear combination,

$$y(x) = C_1y_1(x) + C_2y_2(x),$$

where C_1 and C_2 are arbitrary constants, is also a solution of such homogeneous equation.

All solutions are either exponential functions, or are constructed out of exponential functions. If we try a solution of the form

$$\boxed{y(x) = e^{mx}}$$

where m is a constant, then $y' = me^{mx}$, $y'' = m^2e^{mx}$:

$$\begin{aligned} am^2e^{mx} + bme^{mx} + ce^{mx} &= 0 \\ \Rightarrow\ am^2 + bm + c = 0 \quad &\text{since } e^{mx} \neq 0 \quad \longleftarrow \text{characteristic equation} \\ \Rightarrow \qquad m &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Three cases to be considered (C_1, C_2, C_3, C_4 are constants):

- $b^2 - 4ac > 0\,;$

  Characteristic equation has <u>two distinct real roots</u> m_1 and m_2 giving

  $$\boxed{y(x) = C_1e^{m_1x} + C_2e^{m_2x}}$$

- $b^2 - 4ac < 0\,;$

  Characteristic equation has <u>complex conjugate roots</u> $m_1, m_2 = \alpha \pm i\beta$ where $i = \sqrt{-1}$, giving

  $$\boxed{y(x) = C_3e^{(\alpha+i\beta)x} + C_4e^{(\alpha-i\beta)x}}$$

  Utilising the Euler's formula,

  $$\boxed{e^{\pm i\theta} = \cos\theta \pm i\sin\theta,}$$

solution $y(x)$ can be expressed as

$$\begin{aligned} y(x) &= e^{\alpha x}\Big[C_3 e^{i\beta x} + C_4 e^{-i\beta x}\Big] \\ &= e^{\alpha x}\Big[C_3\big(\cos(\beta x) + i\sin(\beta x)\big) + C_4\big(\cos\beta x - i\sin\beta x\big)\Big] \\ &= e^{\alpha x}\Big[\underbrace{(C_3 + C_4)}_{=C_1}\cos(\beta x) + \underbrace{i(C_3 - C_4)}_{=C_2}\sin(\beta x)\Big] \\ &= e^{\alpha x}\Big[C_1\cos\beta x + C_2\sin\beta x\Big] \end{aligned}$$

- $b^2 - 4ac = 0$;

  Characteristic equation has <u>two equal real roots</u> $m_1 = m_2 = -\dfrac{b}{2a}$ giving

$$\boxed{y(x) = (C_1 x + C_2)e^{m_1 x}}$$

■ EXAMPLE

Solve $2y'' - 5y' - 3y = 0$.

<u>SOLUTION</u>

Solve for the characteristic equation:

$$\begin{aligned} & 2m^2 - 5m - 3 = 0 \\ \Rightarrow\ & (2m+1)(m-3) = 0 \\ \Rightarrow\ & m = -1/2,\ 3 \end{aligned}$$

General solution:

$$y(x) = C_1 e^{-x/2} + C_2 e^{3x}.$$

■ EXAMPLE

Solve $y'' - 10y' + 25y = 0$.

<u>SOLUTION</u>

From the characteristic equation:

$$m^2 - 10m + 25 = (m-5)^2 = 0 \qquad \Rightarrow m_1 = m_2 = 5,$$

we have repeated roots, thus the general solution is

$$y(x) = (C_1 x + C_2)e^{5x}.$$

EXAMPLE

Solve $y'' + y' + y = 0$.

SOLUTION

Characteristic equation:

$$\begin{aligned} m^2 + m + 1 &= 0 \\ \Rightarrow\ m &= \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2} \\ &= \frac{-1 \pm i\sqrt{3}}{2} \\ &= \alpha \pm i\beta. \end{aligned}$$

Identifying $\alpha = -\dfrac{1}{2}$ and $\beta = \dfrac{\sqrt{3}}{2}$, the general solution is

$$y(x) = e^{-x/2}\Big[C_1 \cos\Big(\tfrac{\sqrt{3}}{2}x\Big) + C_2 \sin\Big(\tfrac{\sqrt{3}}{2}x\Big)\Big].$$

EXAMPLE

Solve $y'' - 4y' + 13y = 0$, subject to $y(0) = -1$ and $y'(0) = 2$.

SOLUTION

Characteristic equation:

$$\begin{aligned} m^2 - 4m + 13 &= 0 \\ \Rightarrow\ m &= \frac{4 \pm \sqrt{16 - 52}}{2} \\ &= \frac{4 \pm 6i}{2} \\ &= 2 \pm 3i \end{aligned}$$

General solution:

$$y(x) = e^{2x}\Big[C_1 \cos(3x) + C_2 \sin(3x)\Big]$$

The condition $y(0) = -1$ implies that

$$\begin{aligned} -1 &= e^0(C_1 \cdot 1 + C_2 \cdot 0) \\ &= C_1 \\ \Rightarrow\ y(x) &= e^{2x}\Big[C_2 \sin(3x) - \cos(3x)\Big] \end{aligned}$$

Differentiating this equation with respect to x, and using $y'(0) = 2$ gives

$$\begin{aligned} \frac{dy}{dx} &= e^{2x}\Big[3C_2 \cos(3x) + 3\sin(3x)\Big] + 2e^{2x}\Big[C_2 \sin(3x) - \cos(3x)\Big] \\ \Rightarrow\quad 2 &= e^0(3C_2 + 0) + 2e^0(0 - 1) \\ &= 3C_2 - 2 \end{aligned}$$

$\Rightarrow\ C_2 = 4/3$

Particular solution:

$$y(x) = e^{2x}\left[\frac{4}{3}\sin(3x) - \cos(3x)\right].$$

3.2 Nonhomogeneous Equation

Any function $y_p(x)$, *free of arbitrary parameters*, that satisfies

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x),$$

where a, b, c are constants, and $g(x)$ is continuous is said to be a *particular integral* of the equation.

We solve the nonhomogeneous equation in three steps:

(1) Solve the associated homogeneous equation to get the *complementary function*, denoted by $y_C(x)$,

$$a\frac{d^2y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C = 0.$$

(2) Find the particular integral, $y_P(x)$, of the nonhomogeneous equation.

(3) Add $y_C(x)$ and $y_P(x)$ to form the general solution of the nonhomogeneous equation,

$$y(x) = y_C(x) + y_P(x).$$

When $g(x)$ consists of

(i) a constant k,

(ii) a polynomial in x,

(iii) an exponential function $e^{\alpha x}$,

(iv) trigonometric functions $\sin(\beta x)$ and $\cos(\beta x)$,

or finite sums and products of these functions, it is usually possible to find $y_P(x)$ by the *method of undetermined coefficients*. Other techniques, such as *variation of parameters*, are available for more general $g(x)$.

EXAMPLE

Solve $y'' + 8y' = 24$.

SOLUTION

Solving $y_C'' + 8y_C' = 0$ for the complementary function:

$$\begin{aligned} m^2 + 8m = m(m+8) &= 0 \quad \longleftarrow \text{characteristic equation} \\ \Rightarrow \quad m &= 0, -8 \\ \Rightarrow \quad y_C &= C_1 e^{0\cdot x} + C_2 e^{-8x} \\ &= C_1 + C_2 e^{-8x} \end{aligned}$$

Solving $y_P'' + 8y_P' = 24$ for the particular integral:

$$\begin{aligned} \text{Try} \quad y_P &= A \quad (\text{constant}), \\ y_P' &= 0, \\ y_P'' &= 0, \\ \Rightarrow \quad y_P'' + 8y_P' &= 0 + 0 = 24 \quad \longrightarrow \text{No solution!} \end{aligned}$$

Since there is a constant in y_C, namely C_1, and there is also a constant term (A) in the proposed y_P, this will not work! Instead, multiple y_P by x, and try again,

$$\begin{aligned} \text{Try} \quad y_P &= Ax, \\ y_P' &= A, \\ y_P'' &= 0, \\ \Rightarrow \quad y_P'' + 8y_P' &= 0 + 8A = 24 \\ \Rightarrow \quad A &= 3 \\ \Rightarrow \quad y_P &= 3x. \end{aligned}$$

General solution:

$$\begin{aligned} y(x) &= y_C + y_P \\ &= C_1 + C_2 e^{-8x} + 3x \end{aligned}$$

CHECK:

$$\begin{aligned} y &= C_1 + C_2 e^{-8x} + 3x, \\ y' &= -8C_2 e^{-8x} + 3, \\ y'' &= 64C_2 e^{-8x}, \\ \Rightarrow \quad y'' + 8y' &= 64C_2 e^{-8x} + 8\big(-8C_2 e^{-8x} + 3\big) \\ &= 64C_2 e^{-8x} - 64C_2 e^{-8x} + 24 \end{aligned}$$

$$
\begin{aligned}
&= 24 \\
&= g(x) \quad \text{as required!}
\end{aligned}
$$

Note that if $y_p = Ax$ fail to work, try $y_P = Ax^2$ (multiply y_P by x again). Repeat this step until you can solve for y_P.

EXAMPLE

Determine the solution of $y'' - 2y' - 3y = 8e^{3x}$, subject to $y(0) = y'(0) = 0$.

SOLUTION

Characteristic equation:

$$
\begin{aligned}
& m^2 - 2m - 3 = 0 \\
\Rightarrow\ & (m-3)(m+1) = 0 \\
\Rightarrow\ & m = -1, 3.
\end{aligned}
$$

Complementary function:

$$y_C = C_1 e^{-x} + C_2 e^{3x}.$$

Particular integral:

$$
\begin{aligned}
\text{Try} \quad y_P &= Ae^{3x} \quad \longleftarrow \text{This will not work!} \\
\text{Try} \quad y_P &= Axe^{3x}, \\
y_P' &= Ae^{3x} + 3Axe^{3x} \\
&= A(1+3x)e^{3x}, \\
y_P'' &= A(3)e^{3x} + 3A(1+3x)e^{3x} \\
&= A(6+9x)e^{3x}, \\
\Rightarrow\ y_P'' - 2y_P' - 3y_P &= A(6+9x)e^{3x} - 2A(1+3x)e^{3x} - 3Axe^{3x} \\
&= A(6+9x-2-6x-3x)e^{3x} \\
&= 4Ae^{3x}.
\end{aligned}
$$

Equating $4Ae^{3x} = 8e^{3x}$ yields $A = 2$. Hence, $y_P(x) = 2xe^{3x}$.
General solution:

$$
\begin{aligned}
y(x) &= C_1 e^{-x} + C_2 e^{3x} + 2xe^{3x} \\
&= C_1 e^{-x} + (C_2 + 2x)e^{3x}.
\end{aligned}
$$

Also,

$$y'(x) = -C_1 e^{-x} + 2e^{3x} + 3(C_2 + 2x)e^{3x}$$

$$= -C_1 e^{-x} + (2 + 6x + 3C_2)e^{3x}.$$

Applying condition, $x = 0$, $y = 0$:

$$0 = C_1 + C_2 \quad \Rightarrow \; C_2 = -C_1.$$

Applying condition, $x = 0$, $y' = 0$:

$$0 = -C_1 + (2 + 3C_2) = 2 + 4C_2 \quad \Rightarrow \; C_2 = -\frac{1}{2}, \; C_1 = -C_2 = \frac{1}{2}.$$

Particular solution:

$$y(x) = \frac{1}{2}e^{-x} + \frac{1}{2}\big(4x - 1\big)e^{3x}.$$

EXAMPLE

Solve $y'' - 2y' - 3y = \sin x$.

SOLUTION

Complementary function (from previous example):

$$y_C = C_1 e^{-x} + C_2 e^{3x}.$$

Particular integral:

$$\begin{aligned}
\text{Try} \quad & y_P = A\sin x && \longleftarrow \text{This will not work!} \\
\text{Try} \quad & y_P = A\cos x && \longleftarrow \text{This will not work either!} \\
\text{Try} \quad & y_P = A\sin x + B\cos x && \longleftarrow \text{Need both circular functions to work.} \\
& y_P' = A\cos x - B\sin x, \\
& y_P'' = -A\sin x - B\cos x,
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \; y_P'' - 2y_P' - 3y_P &= -A\sin x - B\cos x - 2(A\cos x - B\sin x) - 3(A\sin x + B\cos x) \\
&= (-4A + 2B)\sin x + (-2A - 4B)\cos x \\
&= \sin x.
\end{aligned}$$

This requires

$$\begin{aligned}
-4A + 2B = 1 \qquad & \Rightarrow B = \tfrac{1}{2} + 2A \\
-2A - 4B = 0 \qquad & \Rightarrow 2A = -4B = -2 - 8A \quad \Rightarrow 10A = -2
\end{aligned}$$

Hence, $A = -1/5$ and $B = 1/10$.

General solution:

$$y(x) = C_1 e^{-x} + C_2 e^{3x} - \frac{1}{5}\sin x + \frac{1}{10}\cos x.$$

EXAMPLE

Solve $y'' + 2y' + 2y = -10xe^x + 5\sin x$.

SOLUTION

Characteristic equation:

$$\begin{aligned} m^2 + 2m + 2 &= 0 \\ \Rightarrow\ m &= \frac{-2 \pm \sqrt{4-8}}{2} \\ &= \frac{-2 \pm 2i}{2} \\ &= -1 \pm i \quad \text{where } i = \sqrt{-1}. \end{aligned}$$

Complementary function:

$$y_C = e^{-x}(C_1 \cos x + C_2 \sin x).$$

Here

$$g(x) = -10xe^x + 5\sin x,$$

and

$$g'(x) = -10e^x - 10xe^x + 5\cos x,$$

indicates that the particular integral should take the following form,

$$y_P(x) = Axe^x + Be^x + C\sin x + D\cos x,$$

where A, B, C and D are constants. Thence.

$$\begin{aligned} y_P' &= Ae^x + Axe^x + Be^x + C\cos x - D\sin x \\ &= (A + Ax + B)e^x + C\cos x - D\sin x, \end{aligned}$$

$$\begin{aligned} y_P'' &= Ae^x + (A + Ax + B)e^x - C\sin x - D\cos x \\ &= (2A + Ax + B)e^x - C\sin x - D\cos x, \end{aligned}$$

$$\begin{aligned} \Rightarrow\ y_P'' + 2y_P' + 2y_P &= (2A + Ax + B + 2A + 2Ax + 2B + 2Ax + 2B)e^x \\ &\quad + (-C - 2D + 2C)\sin x + (-D + 2C + 2D)\cos x \\ &= (4A + 5B)e^x + 5Axe^x + (C - 2D)\sin x + (2C + D)\cos x. \end{aligned}$$

This requires that

$$\begin{aligned} 5A &= -10 && \Rightarrow A = -2, \\ 4A + 5B &= 0 && \Rightarrow 5B = -4A = 8 \quad \Rightarrow B = 8/5, \\ C - 2D &= 5 && \Rightarrow C = 5 + 2D, \\ 2C + D &= 0 && \Rightarrow D = -2C = -10 - 4D \quad \Rightarrow D = -2,\ C = 1. \end{aligned}$$

Particular integral is

$$y_P(x) = -2xe^x + \frac{8}{5}e^x + \sin x - 2\cos x.$$

General solution is

$$y(x) = e^{-x}(C_1 \cos x + C_2 \sin x) + \sin x - 2\cos x + \big(8/5 - 2x\big)e^x.$$

4 Review Questions

[1] Find the general solutions of the following differential equations:

(a) $\dfrac{dy}{dx} - 2xy = 0\,;$

(b) $xyy' = 1 + x\,;$

(c) $y' - 2y = x\,;$

(d) $y' \cos x - y \sin x = \sin x.$

[2] Determine the solution of $y' + 2y = 4$ with $y(0) = 0$.

[3] Find the solution of $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$ by putting $y = vx$.

[4] A body is cooling in surroundings maintained at 10° C. Its temperature θ° C after t minutes is given by

$$\frac{d\theta}{dt} = -k(\theta - 10),$$

where k is a constant. If the temperature of the body is initially 70° C and 10 minutes later is 40° C,

(a) show that $k = \frac{1}{10} \log 2\,;$

(b) find the body's temperature after a further 15 minutes.

[5] An e.m.f. $E(t)$ is applied to an electrical circuit containing a resistance R in series with an inductance L. The current $i(t)$ at time t in the circuit is given by the solution of

$$L\frac{di}{dt} + Ri = E(t),$$

where $i(0) = 0$. Determine $i(t)$ when $E(t) = E_0 \sin(\omega t)$, where E_0 and ω are constants.

[6] Determine the solution of the differential equation,

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2},$$

which satisfies $y(\pi) = 1$.

[7] Determine the general solution of each of the following differential equations:

(a) $y'' - y' - 6y = 0,$

(b) $y'' + 2y' + 10y = 0,$

(c) $y'' - 9y = 0,$

(d) $y'' - 9y' = 0$.

[8] Determine the solution of $y'' - 2y' - 3y = 8e^{3x}$, subject to $y(0) = y'(0) = 3$.

[9] Determine the general solution of each of the following differential equations:

(a) $y'' - y' - 2y = 4\cosh(2x)$,

(b) $y'' + 2y' + y = e^x \cos x$,

(c) $y'' - 3y' + 2y = 6xe^{-x}$.

[10] A constant e.m.f. E_0 volts is applied to a circuit containing, in series, a resistance R ohms, an inductance L henries and a condenser having capacitance C farads. The charge $q(t)$ in the condenser (which is initially uncharged) is given by

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E_0.$$

If the values of resistance, conductance and capacitance take the values, $R = 100$, $L = 1/200$, $C = 10^{-6}$ and $E_0 = 10^3$, show that

$$q(t) = 10^{-3} + e^{-\alpha t}\big(A\cos(\alpha t) + B\sin(\alpha t)\big) \quad \text{where } \alpha = 10^4.$$

[11] A body of unit mass moves along the x-axis under the action of

- a force of magnitude $\omega^2 x$ directed towards the origin,
- a driving force governed by $e^{-\lambda t}\cos(bt)$,
- a frictional resistance defined by $2\lambda\dot{x}$, where $\dot{x}$ is the speed of the body.

The displacement $x(t)$ at time t is given by

$$\ddot{x} + 2\lambda\dot{x} + \omega^2 x = e^{-\lambda t}\cos(bt).$$

If the body starts from rest at the origin, and $\omega > \lambda$, obtain $x(t)$ for the following two cases:

(a) $b = \sqrt{\omega^2 - \lambda^2}$,

(b) $b \neq \sqrt{\omega^2 - \lambda^2}$.

[12] The angular displacement $\theta(t)$ of a rigid body pendulum oscillating about a fixed axis is governed by

$$I\frac{d^2\theta}{dt^2} = -mgh \cdot \sin\theta,$$

where m, g, h and I are constants. For <u>small</u> oscillations (that is, when θ is small), we can use $\sin\theta \approx \theta$. Show that for small oscillations, the motion is simple harmonic in nature with period $2\pi\sqrt{I/(mgh)}$.

[13] An e.m.f. $E_0 \sin(\omega t)$ is applied to an electrical circuit comprising a resistance R, an inductance L and a capacitance C in series. The current $i(t)$ in the circuit at time t satisfies the following differential equation,

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = E_0\omega\cos(\omega t).$$

If $R^2C < 4L$, determine $i(t)$ for the case when $i(0) = 0$ and $\frac{di}{dt}(0) = 0$.

[14] Determine the solution of the differential equation,

$$\frac{dy}{dx} - 3y = e^{5x} + e^{3x}\cos(2x),$$

which satisfies $y(0) = 0$.

[15] (a) Determine the general solution of $y'' - 4y' = 0$.

(b) Using the result from (a), determine the general solutions of

(i) $y'' - 4y' = 8e^{2x}$;

(ii) $y'' - 4y' = 3e^{4x}$;

(iii) $y'' - 4y' = -16x$.

[16] A model for the growth of a population of size N is governed by

$$\frac{dN}{dt} = kN - Q,$$

where k is the difference between the birth and death rates, and Q is the emigration rate. Both k and Q are assumed to be positive constants.

(a) Solve this differential equation subject to the initial condition $N(0) = N_0$.

(b) Show that in the case $Q > kN_0$, the population crashes (*i.e.* $N = 0$) at time

$$t = \frac{1}{k}\log\left(\frac{Q}{Q - kN_0}\right).$$

5 Answers to Review Questions

[1] (a) $y(x) = ce^{x^2}$

(b) $y^2 = 2\log x + 2x + c$

(c) $y(x) = ae^{2x} - \frac{1}{2}x - \frac{1}{4}$

(d) $y(x) = c \cdot \sec x - 1$

[2] $y(x) = 2 - 2e^{-2x}$

[3] $y^2 = x^2 + cx$

[4] $\theta \approx 20.6^\circ$ C

[5] $i(t) = \dfrac{E_0}{\omega^2 L^2 + R^2}\Big(\omega L e^{-Rt/L} + R\sin(\omega t) - \omega L \cos(\omega t)\Big)$

[6] $y(x) = \dfrac{\sin x}{x^2} + 1$

[7] (a) $y(x) = ae^{3x} + be^{-2x}$

(b) $y(x) = e^{-x}\big(a\cos(3x) + b\sin(3x)\big)$

(c) $y(x) = ae^{3x} + be^{-3x}$

(d) $y(x) = a + be^{9x}$

[8] $y(x) = (2x+1)e^{3x} + 2e^{-x}$

[9] (a) $y(x) = ae^{2x} + be^{-x} + \frac{1}{2}e^{-2x} + \frac{2}{3}xe^{2x}$

(b) $y(x) = (ax+b)e^{-x} + \frac{1}{25}e^{x}(3\cos x + 4\sin x)$

(c) $y(x) = ae^{2x} + be^{x} + \big(x + \frac{5}{6}\big)e^{-x}$

[10] Not available.

[11] (a) $x(t) = \dfrac{t}{2b}e^{-\lambda t}\sin(bt)$

(b) $x(t) = \dfrac{e^{-\lambda t}}{\omega^2 - \lambda^2 - b^2}\Big[\cos(bt) - \cos\big(\sqrt{\omega^2 - \lambda^2}\, t\big)\Big]$

[12] Not available.

[13] Steady-state solution is

$$\frac{E_0 \omega C^2}{(1-\omega^2 LC)^2 + (\omega RC)^2}\left[\omega R\sin(\omega t) + \frac{1}{C}\big(1 - \omega^2 LC\big)\cos(\omega t)\right]$$

[14] $y(x) = \frac{1}{2} + \frac{1}{2}e^{5x} + \frac{1}{2}e^{3x}\sin(2x)$

[15] (a) $y(x) = a + be^{4x}$

(b) (i) $y(x) = a + be^{4x} - 2e^{2x}$

(ii) $y(x) = a + \left(b + \frac{3}{4}x\right)e^{4x}$

(iii) $y(x) = a + be^{4x} + 2x^2 + x$

[16] (a) $N(t) = Q/k + \left(N_0 - Q/k\right)e^{kt}$

(b) Put $N = 0$ and solves for t, noting that $kN_0 - Q < 0$.